Introduction to Hydrodynamics II

Kevin Dusling

BROOKHAVEN NATIONAL LABORATORY

June 8, 2010

School of High Energy Dynamics in Heavy Ion Collisions
Berkeley, California

Contents

1. Introduction: The need for second order hydrodynamics
    - Diffusion Equation
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Navier Stokes

1. Yesterday we looked at NS in 0+1 D: $$\frac{de}{d\tau} = -\frac{e + p - \frac{4}{3}\frac{\eta}{\tau}}{\tau}$$

2. We would like to solve this in 2+1 D,

$$T^{\mu\nu} = T_0^{\mu\nu} - \eta\sigma^{\mu\nu} \qquad\qquad \partial_\mu T^{\mu\nu} = 0$$

   but it turns out there are some problems

   - Instabilities
   - Violations of Causality

3. In order to investigate this we will look at a much simpler theory

Diffusion Equation

1. Continuity Equation

$$\partial_t n + \nabla_i j_i = 0$$

2. $+$ Fick's Law

$$j_i(\mathbf{x}, t) = -D \nabla_i n(\mathbf{x}, t)$$

3. $=$ Diffusion Eqn.

$$\left(\partial_t - D\nabla^2\right) n = 0$$

Diffusion Equation

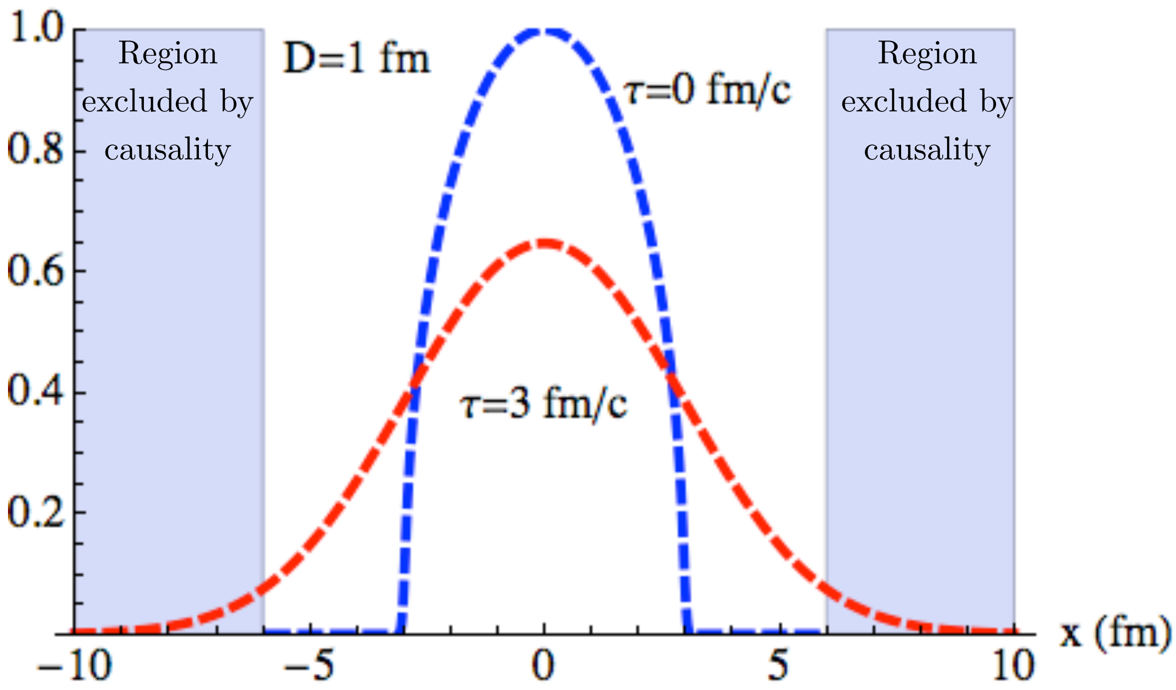
1. Diffusion eqn. in 1+1D

$$\begin{aligned} \left(\partial_t - D\partial_x^2\right) n &= 0 \\ \text{I.C.: } n(x, t=0) &= \phi(x) \end{aligned}$$

2. Solution

$$n(x,t) = \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{+\infty} \phi(y) \exp\left[-\frac{(x-y)^2}{4Dt}\right] dy$$

Diffusion equation in 1+1 D



Telegraph Equation

1. Continuity Equation

$$\partial_t n + \nabla_i j_i = 0$$

2. $+$ Modified Fick's Law

$$j_i(\mathbf{x}, t) + D\nabla_i n(\mathbf{x}, t) = -\tau_R \frac{\partial j_i}{\partial t}$$

3. $=$ Telegraph Eqn.

$$\left(\partial_t - D\nabla^2\right) n = -\tau_R \partial_t^2 n$$

Telegraph Equation

1. Exercise: Find analytic solution to telegraph equation

$$\left(\partial_t - D\nabla^2\right) n = -\tau_R \partial_t^2 n$$

   with the following initial conditions

$$\begin{aligned} n(x, t=0) &= \phi(x) \\ \partial_t n(x, t=0) &= \psi(x) \end{aligned}$$

2. Answer:

$$\begin{aligned} 2e^{t/2\tau_R} n(x,t) &= \phi(x+vt) + \phi(x-vt) \\ &+ \frac{t}{2\tau_R} \int_{x-vt}^{x+vt} \phi(y) \frac{I_1\left(\frac{t}{2\tau_R}\sqrt{(vt)^2 - x^2}\right)}{\sqrt{(vt)^2 - x^2}} \\ &+ \frac{1}{2\tau_R v} \int_{x-vt}^{x+vt} \left[\phi(y) + 2\tau_R \psi(y)\right] I_0\left(\frac{t}{2\tau_R}\sqrt{(vt)^2 - x^2}\right) \end{aligned}$$

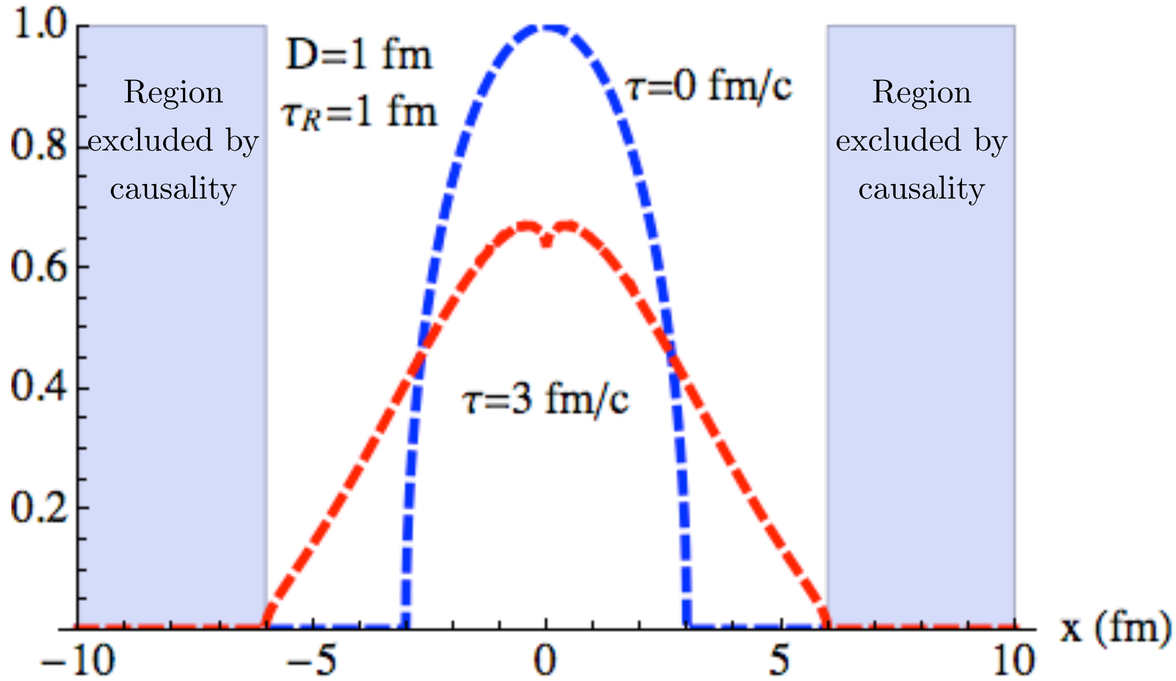
Useful Integrals

1. In case you really try to work this out you will need these integrals

$$\frac{i}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}\frac{e^{-i\tau\sqrt{k^2-a^2}}}{\sqrt{k^2-a^2}} = I_0\left(a\sqrt{\tau^2-x^2}\right)\theta\left(\tau-x\right)$$

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}e^{-i\tau\sqrt{k^2-a^2}} &= I_0\left(a\sqrt{\tau^2-x^2}\right)\delta\left(\tau-x\right)\\ &+ a\tau\frac{I_1\left(a\sqrt{\tau^2-x^2}\right)}{\sqrt{\tau^2-x^2}}\theta\left(\tau-x\right)\end{aligned}$$

Telegraph equation in 1+1 D



1. So it turns out that the proposed second order theory solves our problem of causality

2. wave front propagates out at $v = \sqrt{\dfrac{D}{\tau_R}}$

Coming Back to the NS equations

1. Exercise: Recast the NS equation $\partial_\mu T^{\mu\nu} = 0$ where

$$T^{\mu\nu} = T_0^{\mu\nu} - \eta\sigma^{\mu\nu}$$

$$\sigma^{\mu\nu} = \nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3}\Delta^{\mu\nu}\nabla_\lambda u^\lambda$$

into the following form:

$$\begin{aligned} De + (e+p)\,\nabla_\mu u^\mu &= \frac{\eta}{2}\sigma^{\mu\nu}\sigma_{\mu\nu} \\ Du^\mu + \frac{\nabla^\mu p}{e+p} &= \frac{1}{(e+p)}\Delta^\mu_\alpha \partial_\beta \left(\eta\sigma^{\alpha\beta}\right) \end{aligned}$$

Linearized NS equations

1. Let's perform a linearized analysis of the NS equations

$$\begin{aligned} De + (e+p)\,\nabla_\mu u^\mu &= \frac{\eta}{2}\sigma^{\mu\nu}\sigma_{\mu\nu} \\ Du^\mu + \frac{\nabla^\mu p}{e+p} &= \frac{1}{(e+p)}\Delta^\mu_\alpha \partial_\beta\left(\eta\sigma^{\alpha\beta}\right) \end{aligned}$$

2. Start by perturbing the energy density and flow velocity

$$e(t,\mathbf{x}) = e_0 + \delta e(t,y)$$

$$u^\mu = (1,\mathbf{0}) + \delta u^\mu(t,y)$$

Linearized NS equations

1. The linearized NS equations reduce to a diffusion equation

$$\partial_t \delta u^z - \frac{\eta}{(e_0 + p_0)} \partial_y^2 \delta u^z = 0 \qquad\qquad \left(\partial_t - D\nabla^2\right) n = 0$$

2. Let us consider a sinusoidal perturbation

$$\delta u^z(t, y) \propto e^{\omega t - iky}$$

3. We find a "dispersion relation" of the form

$$\omega = \frac{\eta}{(e_0 + p_0)} k^2$$

4. so we can estimate the speed of a diffusion mode with wavenumber k

$$v(k) = \frac{d\omega}{dk} = 2\frac{\eta}{(e_0 + p_0)} k$$

Linearized NS equations

1. Let us modify the NS equations in the same was as in the diffusion case

$$\partial_t \delta u^z - \frac{\eta}{(e_0 + p_0)} \partial_y^2 \delta u^z = -\tau_R \partial_t^2 \delta u^z \qquad \left(\partial_t - D\nabla^2\right) n = -\tau_R \partial_t^2 n$$

2. Considering again a sinusoidal perturbation

$$\delta u^z(t, y) \propto e^{i\omega t - iky}$$

3. Exercise: Show the diffusion speed at large k is finite and

$$\lim_{k \to \infty} \frac{d\omega}{dk} = \sqrt{\frac{\eta}{\tau_R (e + p)}}$$

BRSSS stress energy tensor

1. BRSSS wrote down all possible second order gradients allowed by conformal invariance

$$
\begin{aligned}
\pi^{\mu\nu} &= -\eta\sigma^{\mu\nu} + \eta\tau_\pi \left[\langle D\sigma^{\mu\nu} \rangle + \frac{1}{d-1}\sigma^{\mu\nu}\partial\cdot u \right] \\
&+ \lambda_1 \langle \sigma^{\mu}_{\lambda}\sigma^{\nu\lambda} \rangle + \lambda_2 \langle \sigma^{\mu\lambda}\Omega^{\nu\lambda} \rangle + \lambda_3 \langle \Omega^{\mu}_{\lambda}\Omega^{\nu\lambda} \rangle
\end{aligned}
$$

where the vorticity is defined as

$$
\Omega^{\mu\nu} \equiv \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\partial_\alpha u_\beta - \partial_\beta u_\alpha\right)
$$

BRSSS stress energy tensor

1. The equations of motion are

$$T^{\mu\nu} = T^{\mu\nu}_{\text{ideal}} + \pi^{\mu\nu} \qquad\qquad \partial_\mu T^{\mu\nu} = 0$$

where $\pi^{\mu\nu}$ has been promoted to a dynamical variable evolving according to

$$\begin{aligned} \pi^{\mu\nu} &= -\eta\sigma^{\mu\nu} - \tau_\pi \left[\langle D\pi^{\mu\nu} \rangle + \frac{d}{d-1}\pi^{\mu\nu}\partial \cdot u \right] \\ &+ \frac{\lambda_1}{\eta^2}\langle \pi^\mu_\lambda \pi^{\nu\lambda} \rangle - \frac{\lambda_2}{\eta}\langle \pi^{\mu\lambda}\Omega^{\nu\lambda} \rangle + \lambda_3 \langle \Omega^\mu_\lambda \Omega^{\nu\lambda} \rangle \end{aligned}$$

Recap: zeroth order solution

1. Yesterday, we found the zeroth order solution to the Boltzmann eqn.

$$\left(\partial_t + v^i_{\mathbf{p}}\partial_i\right) f(\mathbf{p}, \mathbf{x}, t) = -\mathcal{C}[f, \mathbf{p}]$$

$$\mathcal{L}f \;=\; \frac{1}{\epsilon}\mathcal{C}[f, \mathbf{p}]$$

2. We expanded in terms of ε

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \cdots$$

$$\mathcal{C}[f_0, \mathbf{p}] = 0 \quad \longrightarrow \quad f^0(P, X) = \exp\left(\frac{p^\mu u_\mu - \mu}{T}\right)$$

First order solution

1. In operator notation, f_1 is the solution to the following intego-differential equation

$$\mathcal{L} f_0 = \mathcal{C}[f_1|f_0, \mathbf{p}] + \mathcal{C}[f_0|f_1, \mathbf{p}]$$

2. where the collision operator is

$$\mathcal{C}[f, g, \mathbf{p}] = \frac{1}{p} \int_{\mathbf{q}} \int_{\mathbf{q}'} \int_{\mathbf{p}'} |\mathcal{M}|^2 (2\pi)^4 \delta^4 \left(P + Q - P' - Q'\right) \left[f_{\mathbf{q}'} g_{\mathbf{p}'} - f_{\mathbf{q}} g_{\mathbf{p}}\right]$$

Left hand side

1. We first need to evaluate

$$\mathcal{L} f_0 \equiv \left(\partial_t + v^i_{\mathbf{p}} \partial_i\right) f_0(\mathbf{p}, \mathbf{x}, t) \equiv \frac{p^\mu}{E_{\mathbf{p}}} \partial_\mu f_0(\mathbf{p}, \mathbf{x}, t)$$

$$\frac{p^\mu}{E_{\mathbf{p}}} \partial_\mu \exp\left(\frac{p^\alpha u_\alpha}{T}\right) = f_0 \left[\frac{p^\mu p^\alpha \partial_\mu u_\alpha}{E_{\mathbf{p}} T} + p^\mu \partial_\mu \left(\frac{1}{T}\right)\right]$$

Left hand side

1. Exercise: Show for a conformal theory that

$$\frac{p^\mu p^\alpha \partial_\mu u_\alpha}{E_{\mathbf{p}} T} + p^\mu \partial_\mu \left(\frac{1}{T} \right) = \frac{p^\mu p^\alpha \sigma_{\mu\alpha}}{2 E_{\mathbf{p}} T}$$

2. And therefore

$$\mathcal{L} f_0 = f_0 \frac{1}{2 E_{\mathbf{p}} T} p^\mu p^\nu \sigma_{\mu\nu}$$

Relaxation time approximation

1. Let us use a very simplistic model for the collision operator

$$\mathcal{L} f_0 = \mathcal{C}_{\mathrm{RT}}[f_1, \mathbf{p}]$$

where $\mathcal{L} f_0 = f_0 \dfrac{1}{2E_{\mathbf{p}}T} p^\mu p^\nu \sigma_{\mu\nu}$

and $\mathcal{C}_{\mathrm{RT}}[f, \mathbf{p}] = -\dfrac{f(p) - f_0(p)}{\tau_R(E_p)}$

2. And we can solve for f_1

$$f_1 - f_0 \equiv \delta f = -f_0 \frac{\tau_R(E_{\mathbf{p}})}{2E_{\mathbf{p}}T} p^\mu p^\nu \sigma_{\mu\nu}$$

Relaxation time approximation

1. The relaxation time sets the shear viscosity

2. Exercise: Starting with the definition of the stress-energy tensor

$$T^{ij} \equiv p\delta^{ij} - \eta\langle\partial^i u^j\rangle = \int_{\mathbf{p}} \frac{p^i p^j}{E_p}\; f_o + \delta f(p)$$

and the form of df we just worked out

$$\delta f = -f_0 \frac{\tau_R(E_{\mathbf{p}})}{2E_{\mathbf{p}}T} p^\mu p^\nu \sigma_{\mu\nu}$$

get the following relation between the relaxation time and shear viscosity

$$\eta = \frac{1}{30T}\int_{\mathbf{p}} E_{\mathbf{p}}^2\; f_o\; \tau_R(E_{\mathbf{p}})$$

Summary

1. We have our 2nd order equations of motion

$$T^{\mu\nu} = T^{\mu\nu}_{\text{ideal}} + \pi^{\mu\nu} \qquad\qquad \partial_\mu T^{\mu\nu} = 0$$

$$\pi^{\mu\nu} \quad = \quad -\eta\sigma^{\mu\nu} - \tau_\pi \langle D\pi^{\mu\nu}\rangle + \cdots$$

2. And we know what is going on at the level of Kinetic Theory

$$\delta f = -f_0 \frac{\tau_R(E_{\mathbf{p}})}{2E_{\mathbf{p}}T} p^\mu p^\nu \sigma_{\mu\nu}$$

3. So now we can go and solve

Elements of a hydrodynamic simulation

1. Initial Conditions

2. Solving

3. Freeze-out

Initial Conditions

1. The initial conditions are really outside the realm of hydrodynamics

2. But in order to solve we need to specify

$$T(\mathbf{x}_\perp, \tau_0), u^\mu(\mathbf{x}_\perp, \tau_0), \pi^{\mu\nu}(\mathbf{x}_\perp, \tau_0)$$

3. Two of these are "easy"

$$u^\mu(\mathbf{x}_\perp, \tau_0) = 0$$

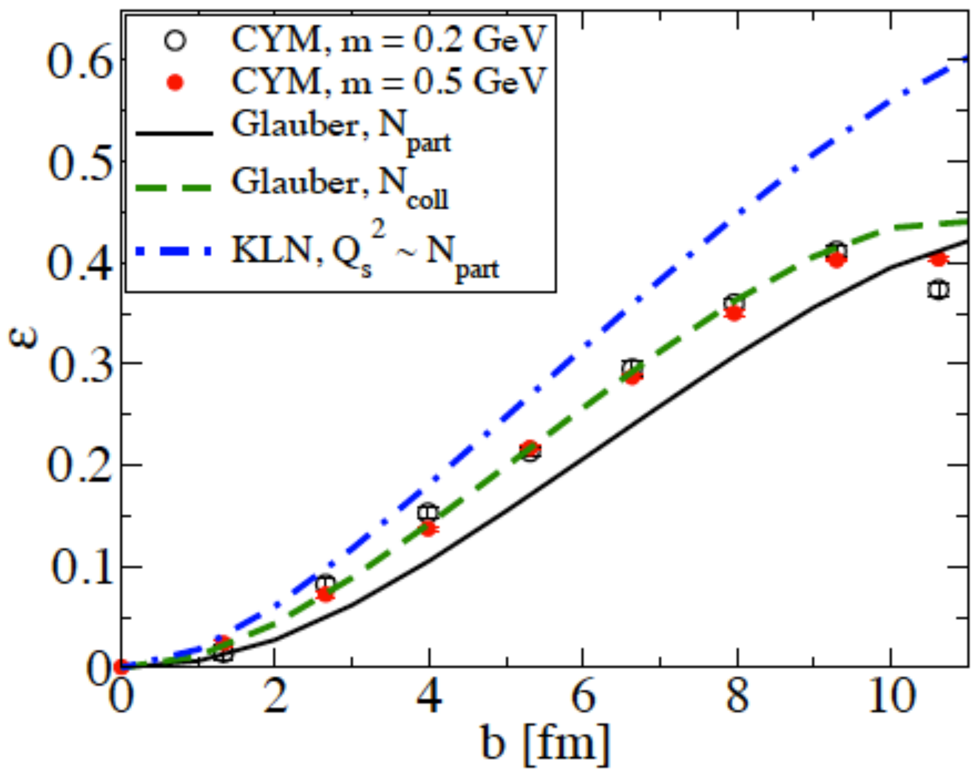
$$\pi^{\mu\nu}(\mathbf{x}_\perp, \tau_0) = -\eta\sigma^{\mu\nu} = \mathrm{diag}(0, +\frac{2\eta}{3\tau}, +\frac{2\eta}{3\tau}, -\frac{4\eta}{3\tau})$$

4. What really controls everything is the energy density

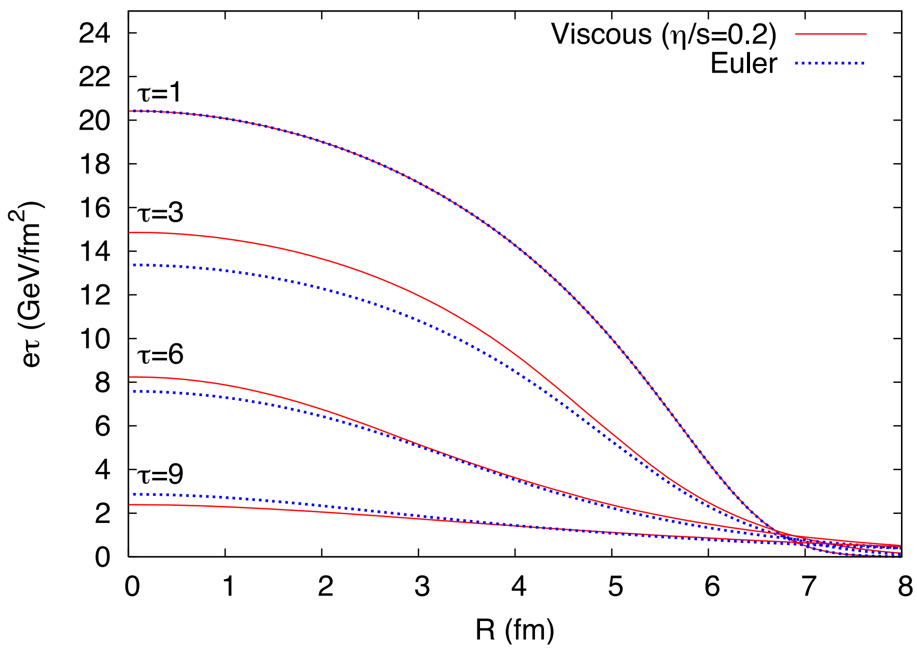
$$e(\mathbf{x}_\perp, \tau_0) \quad \text{or} \quad T(\mathbf{x}_\perp, \tau_0) \quad \text{or} \quad s(\mathbf{x}_\perp, \tau_0)$$

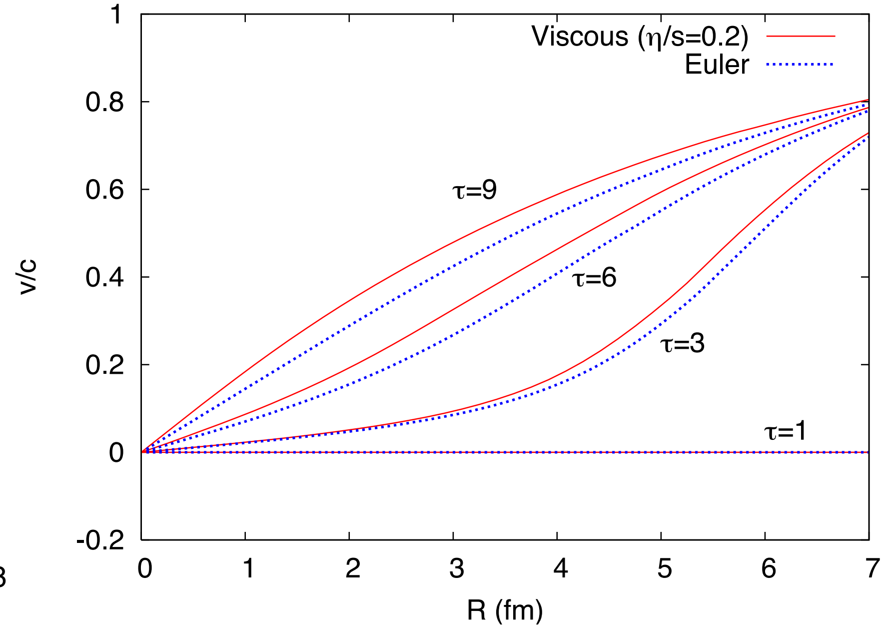
Glauber Theory

1. The assumption is that the collision of two nuclei can be described by the incoherent superposition of an equivalent number of nucleon-nucleon collisions



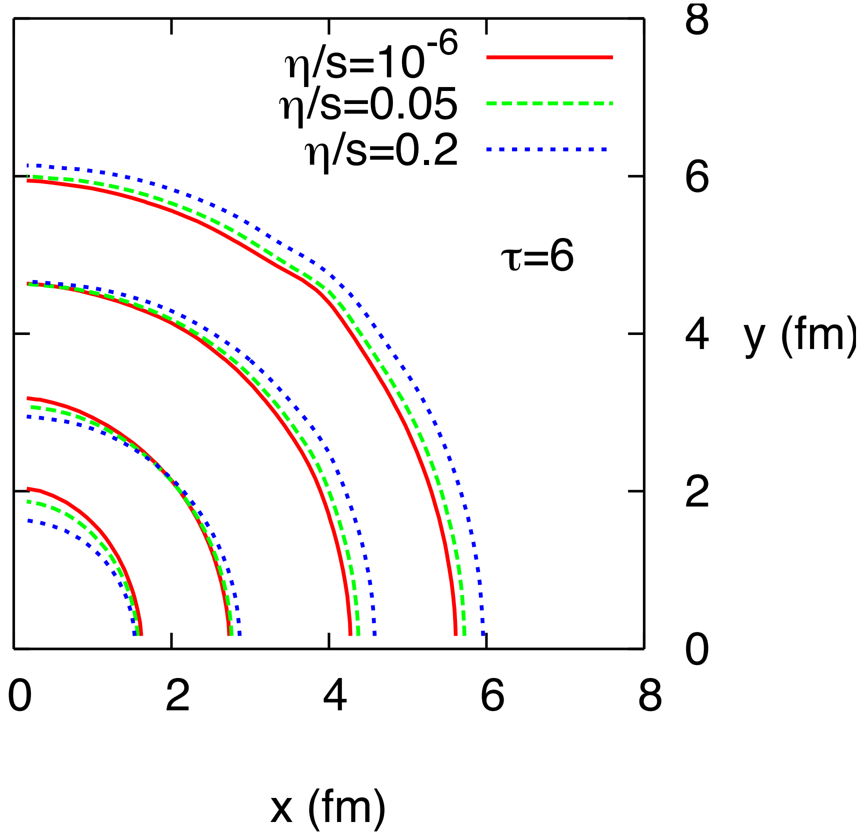
1+1 D



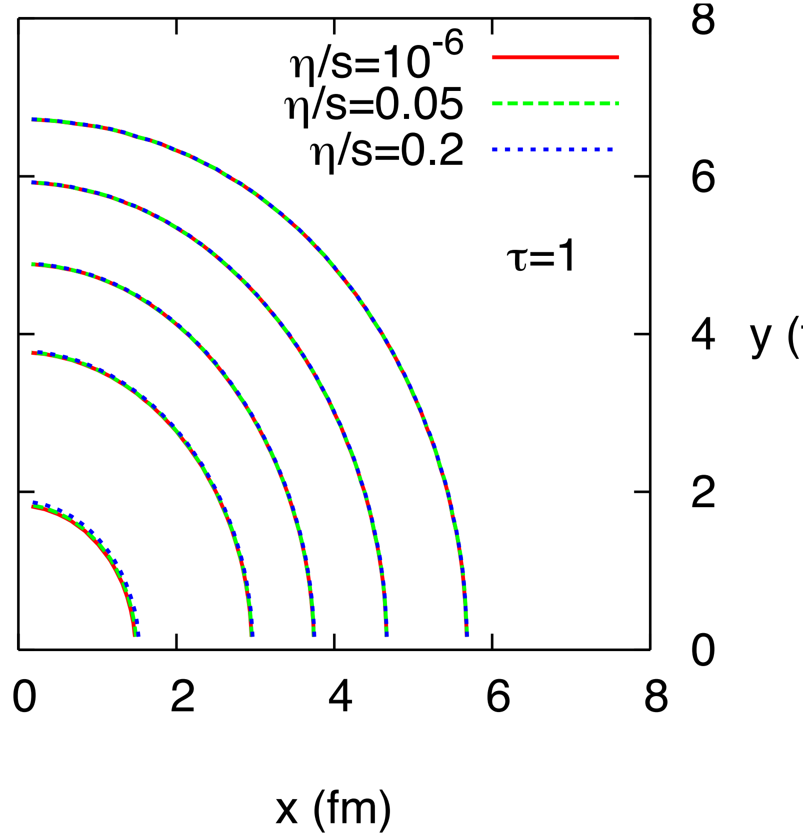


1. The longitudinal pressure is initially lower in the viscous case
2. Less pdV work is done so the energy density depletes slower in viscous case
3. The larger transverse expansion at later times causes a quicker depletion of the energy density at later times

2+1 D



η/s=10^-6
η/s=0.05
η/s=0.2
τ=6
0
2
4
6
8
0
2
4
6
8
y (fm)
x (fm)

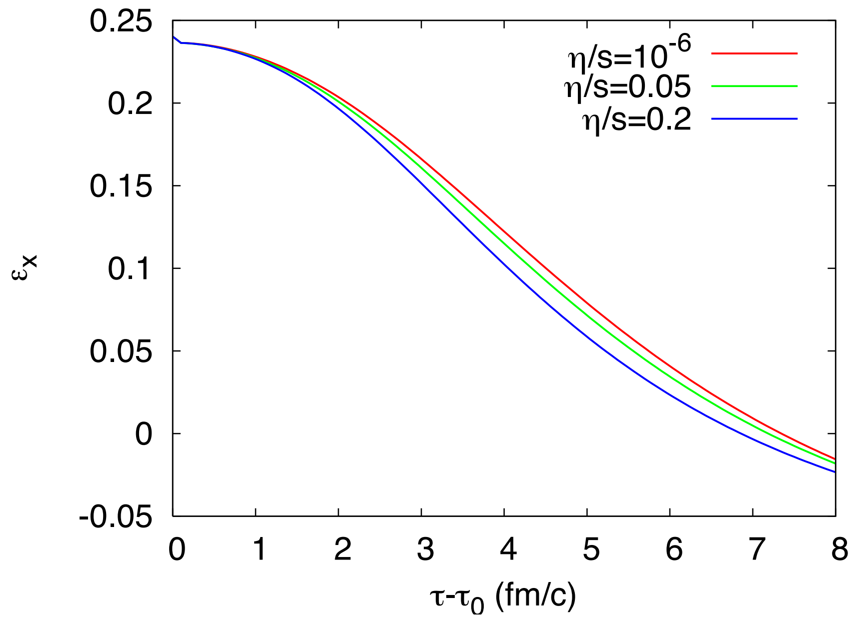


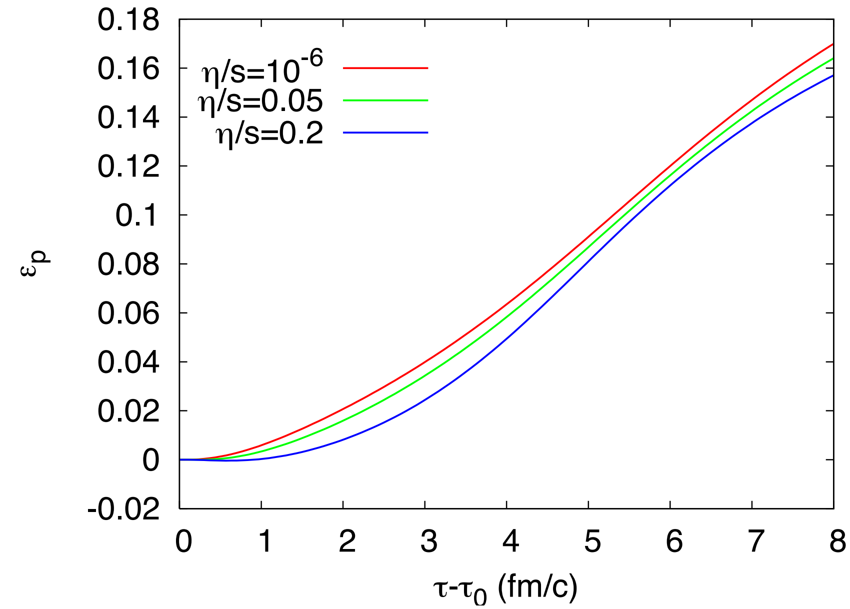
η/s=10^-6
η/s=0.05
η/s=0.2
τ=1
0
2
4
6
8
0
2
4
6
8
y (f
x (fm)

2+1 D

$$\epsilon_x = \frac{\langle\langle y^2 - x^2 \rangle\rangle}{\langle\langle y^2 + x^2 \rangle\rangle}$$

$$\epsilon_p = \frac{\langle\langle T_{xx} - T_{yy} \rangle\rangle}{\langle\langle T_{xx} + T_{yy} \rangle\rangle}$$





Freeze-out

1. Ultimately experiments measure particle and it is necessary to convert the hydrodynamic information $T(\mathbf{x}), u^\mu(\mathbf{x}), \pi^{\mu\nu}(\mathbf{x})$ we just solved for into particle spectra

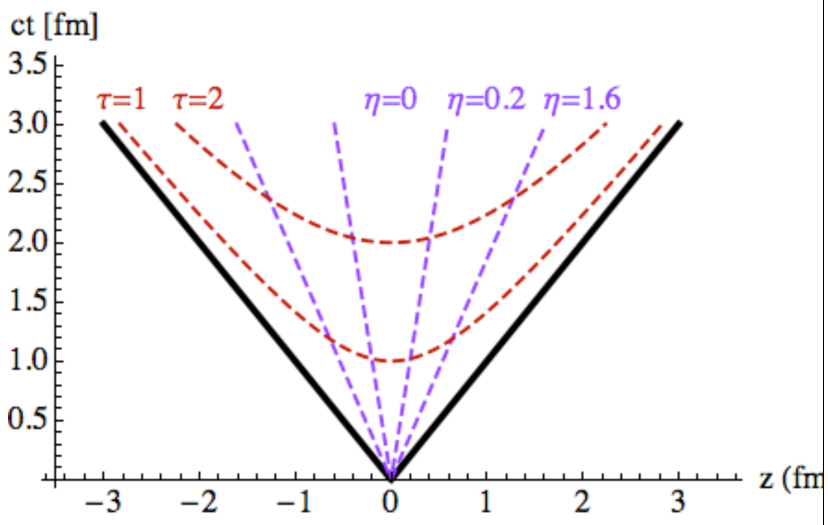
2. This is done using the "Cooper-Frye" formula

$$E\frac{d^3N}{d^3\mathbf{p}} = \frac{1}{(2\pi)^3}\int_\Sigma d\Sigma_\mu P^\mu f(P,X)$$

Freeze-out

1. As an example lets freeze-out at fixed proper time

$$d\Sigma_\mu = (dV, 0, 0, 0)$$



and we get the following result

$$E\frac{d^3N}{d^3\mathbf{p}} = \frac{1}{(2\pi)^3}\int \tau d\eta d^2\mathbf{x}_\perp p^0 f(P,X)$$

Freeze-out

1. Typically one chooses a freeze-out hyper-surface of constant temperature or energy density

2. In order to understand viscous corrections lets take the following alternative.

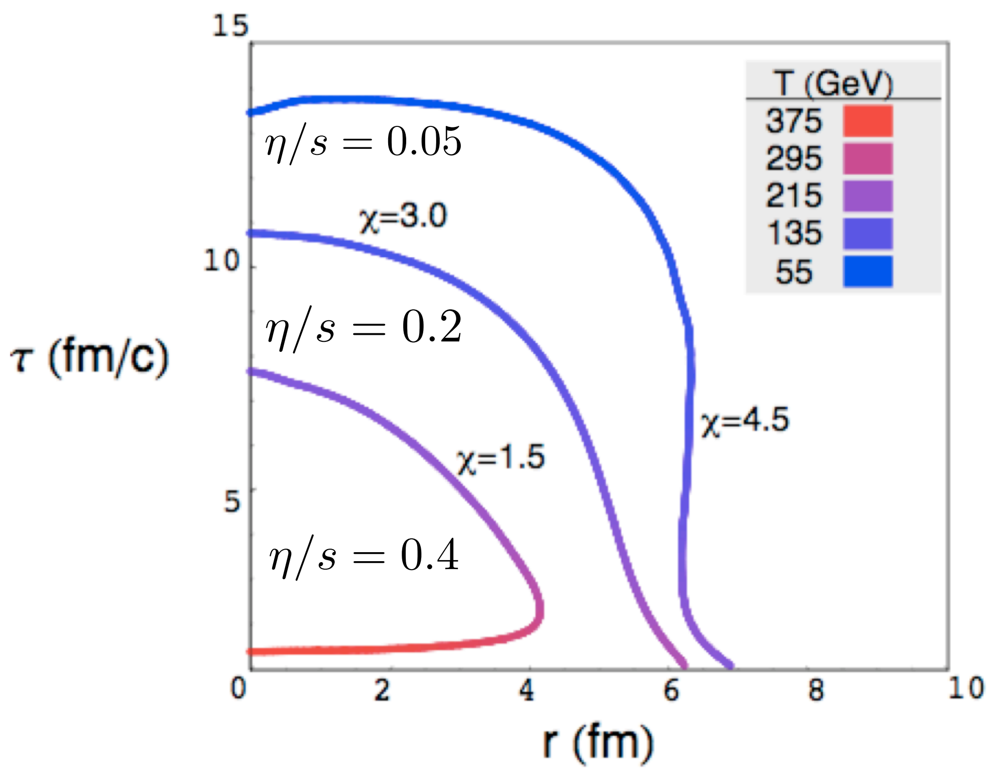
3. Yesterday we specified when hydrodynamics was applicable in 0+1 D

$$\frac{\eta}{e+p}\frac{1}{\tau} \ll 1$$

4. The expansion rate in 3+1 D is $\nabla_\mu u^\mu$

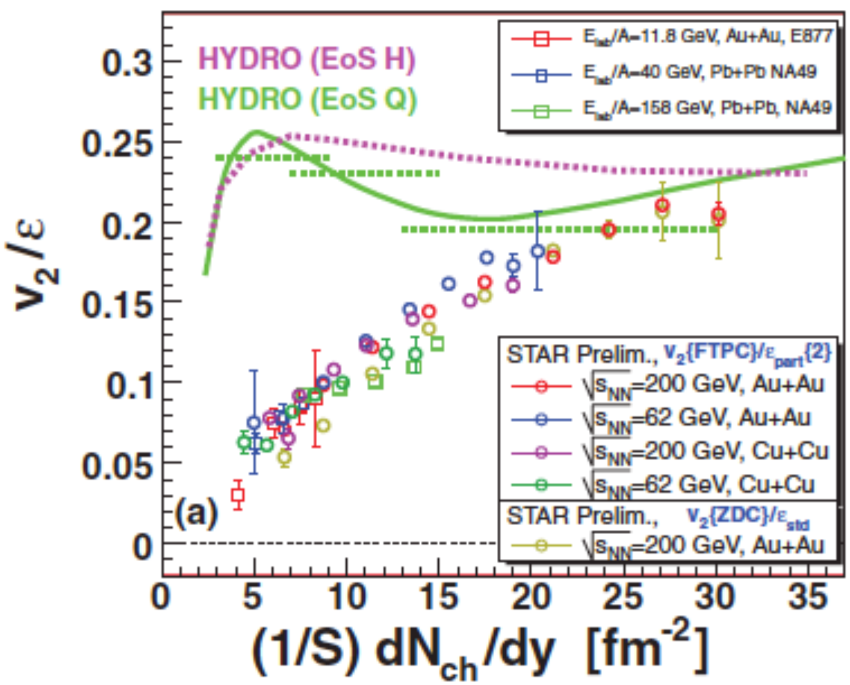
Freeze-out

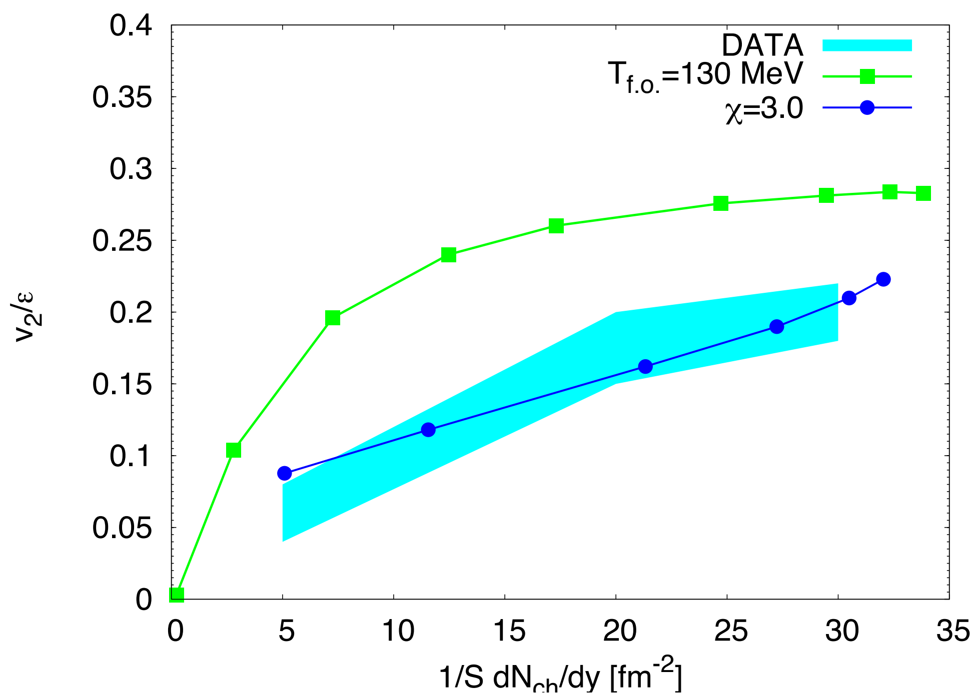
1. Let's freeze-out on contours of constant $\dfrac{\eta}{p}\partial_\mu u^\mu \sim \tau_R \partial_\mu u^\mu$



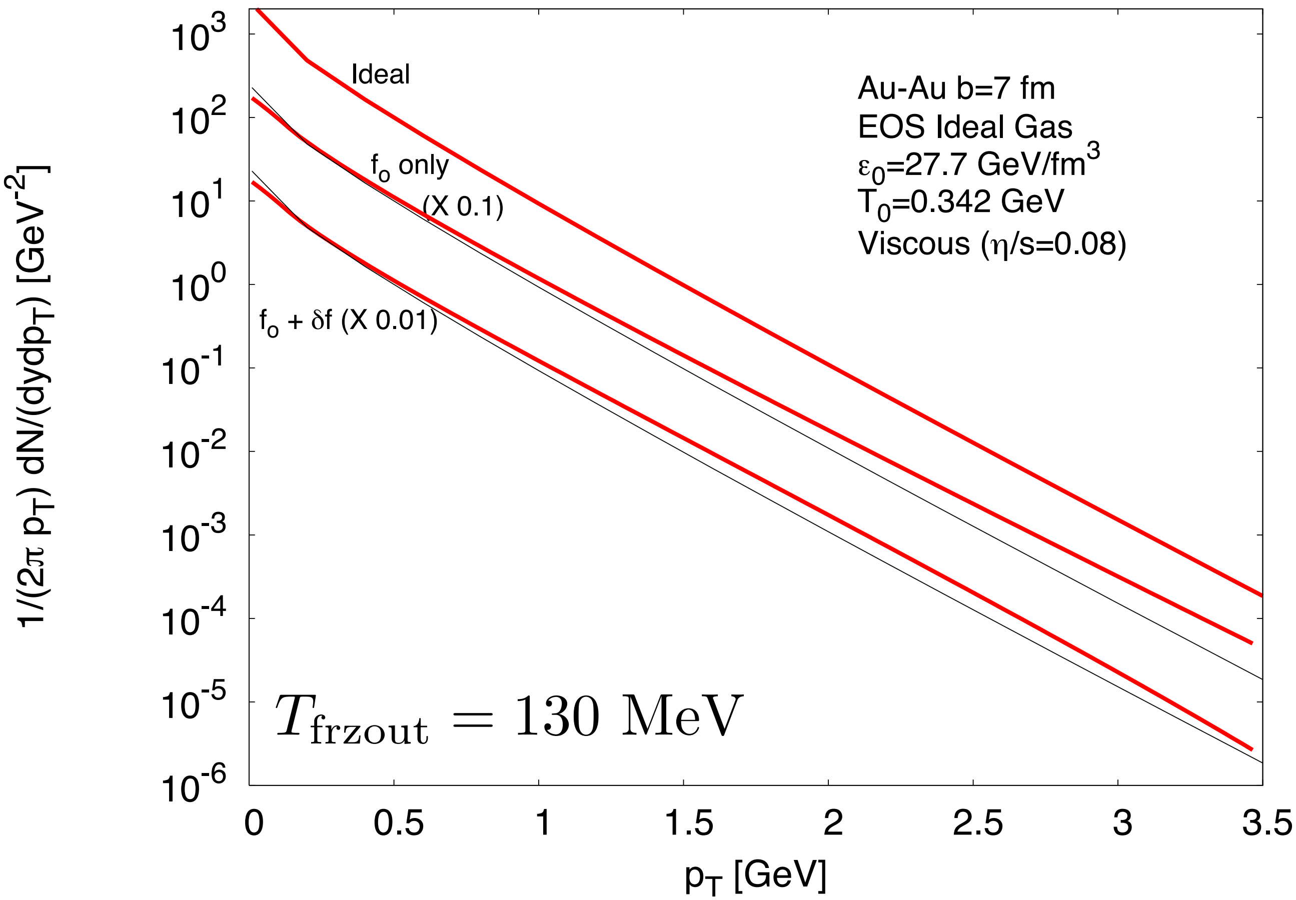
Freeze-out

1. Viscosity sets the necessary scale for freeze-out
2. And can possibly help us understand multiplicity scaling





Viscous correction to spectra



How does viscosity manifest itself in spectra?

1. Viscous correction to equation of motion

$$\partial_\mu T^{\mu\nu} = 0 \qquad \text{where} \qquad T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + pg^{\mu\nu} - \eta\langle\partial^\mu u^\nu\rangle$$

2. Viscous correction to spectra

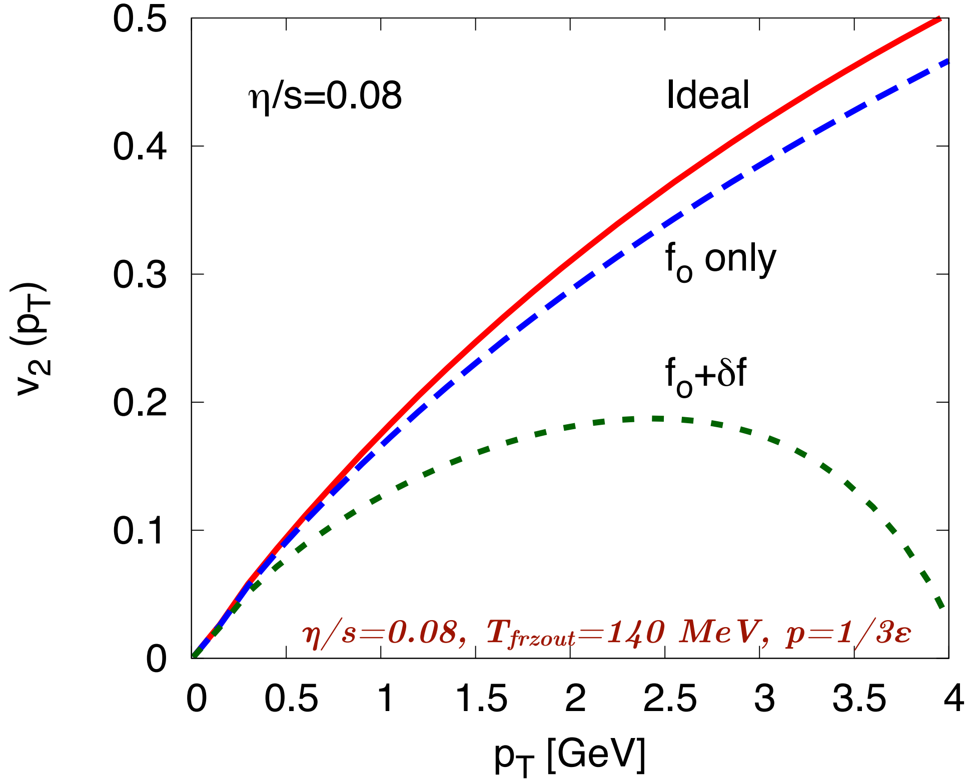
$$E\frac{d^3N}{d^3p} = \frac{\nu}{(2\pi)^3}\int_\sigma f_o + \delta f \; p^\mu d\sigma_\mu$$

$$\delta f = -\frac{\eta}{sT^3} \times f_0(p)p^i p^j \langle\partial_i u_j\rangle$$

3. In the above expression we have taken what is called the "quadratic ansatz" for the off-equilibrium distribution function corresponding to

$$\tau_R \propto E_p$$

How does viscosity manifest itself in spectra?



We need to have a quantitative understanding
of δf and quadratic ansatz.

Reminder

1. We started with the Boltzmann equation in the RTA

$$\partial_t f + v_{\mathbf{p}} \cdot \partial_{\mathbf{x}} f = -\frac{f(p) - f_0(p)}{\tau_R(E_p)}$$

Substitute $f(p) = f_o(p) + \delta f(p)$ and find

$$\delta f \propto \frac{\tau_R(E_p)}{E_p} f_0(p) p^i p^j \langle \partial_i u_j \rangle$$

2. We just showed results for the quadratic ansatz $\tau_R \propto E_p$
but what about $\tau_R \propto (E_p)^\beta$?

Notation

1. Most general form of off equilibrium correction is

$$\delta f = -\chi(\tilde{p}) \times f_0 \hat{p}^i \hat{p}^j \langle \partial_i u_j \rangle$$

where $\tilde{p} \equiv \dfrac{p}{T}$ and $\hat{p}^i \equiv \dfrac{p^i}{|\mathbf{p}|}$

Two Extreme Limits

1. Quadratic: Relaxation time growing with energy

$$\tau_R \propto E_p \qquad \frac{dp}{dt} \propto \text{const.} \qquad \chi(p) \propto p^2$$

2. Linear: Relaxation time independent of Parton energy

$$\tau_R \propto \text{const.} \qquad \frac{dp}{dt} \propto p \qquad \chi(p) \propto p$$

3. As we will show reality is somewhere in between

Connection between δf and viscosity

$$T^{ij} \equiv p\delta^{ij} - \eta\langle\partial^i u^j\rangle = \int_{\mathbf{p}} \frac{p^i p^j}{E_p}\, f_o + \delta f(p)$$

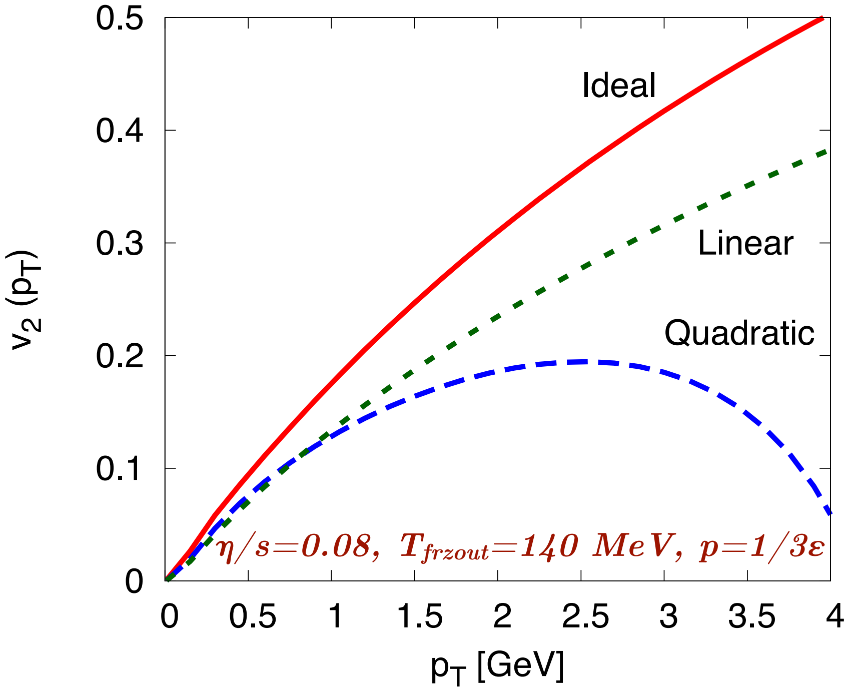
First moment of δf determines shear viscosity.

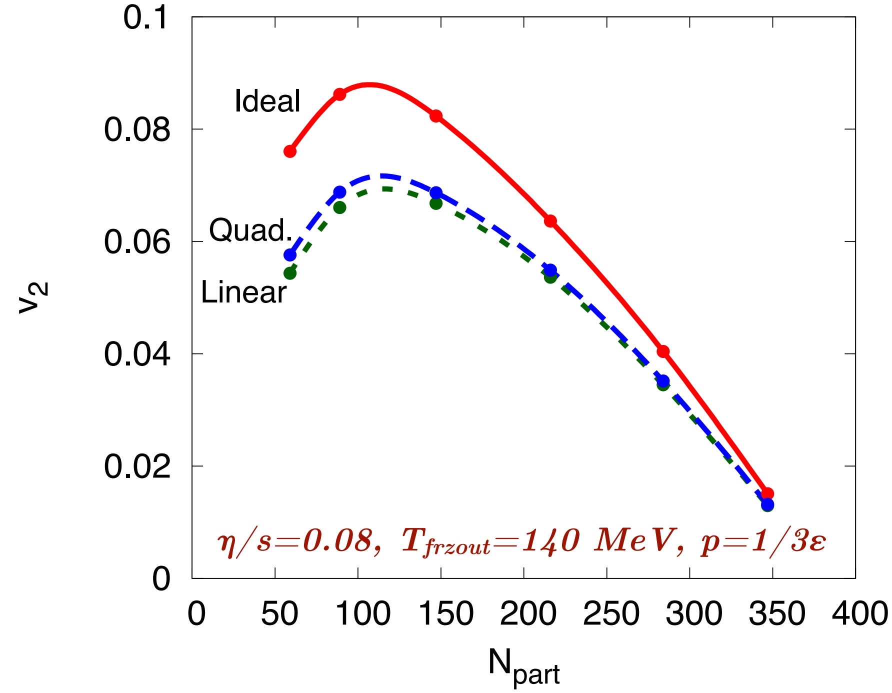
$$\delta f = -\chi(\tilde{p}) \times f_0 \hat{p}^i \hat{p}^j \langle\partial_i u_j\rangle \longrightarrow \eta = \frac{1}{15}\int_{\mathbf{p}} f_o \chi(p) p$$

$$\chi(\tilde{p}) = \frac{120}{\Gamma(6-\alpha)} \times \frac{\eta}{sT} \times \tilde{p}^{2-\alpha}$$

So the form of δf is partially constrained by viscosity.

Two Extreme Limits





$$\eta \propto \int_{\mathbf{p}} p\ f_0 \chi(p)$$

$$\delta \overline{v_2} \propto \int_{\mathbf{p}} p^2\ f_0 \chi(p)$$

Weakly coupled pure-glue QCD

1. Boltzmann equation

$$\partial_t f + v_{\mathbf{p}} \cdot \partial_{\mathbf{x}} f = -\mathcal{C}^{2\leftrightarrow 2}[f] - \mathcal{C}^{1\leftrightarrow 2}[f]$$

2. Substitute $f(p) = f_o(p) + \delta f(p)$ and find

$$f_o \frac{p^i p^j}{T E_p} \langle \partial_i u_j \rangle = -\mathcal{C}^{2\leftrightarrow 2}[\delta f] - \mathcal{C}^{1\leftrightarrow 2}[\delta f]$$

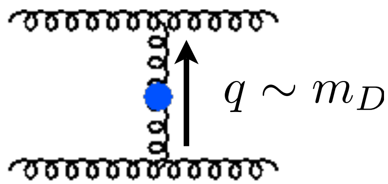
3. This integral equation can be inverted to obtain δf.

Weakly coupled pure-glue QCD

1. Three different modes of energy loss

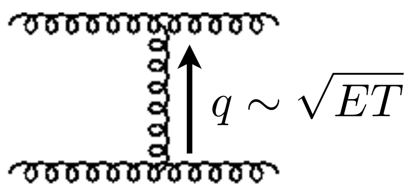
Asymptotic Forms

1. Soft Scattering



$$\frac{dp}{dt} \propto g^4 \log\left(\frac{T}{m_D}\right) \qquad \chi(p) \propto p^2$$

2. Collisional



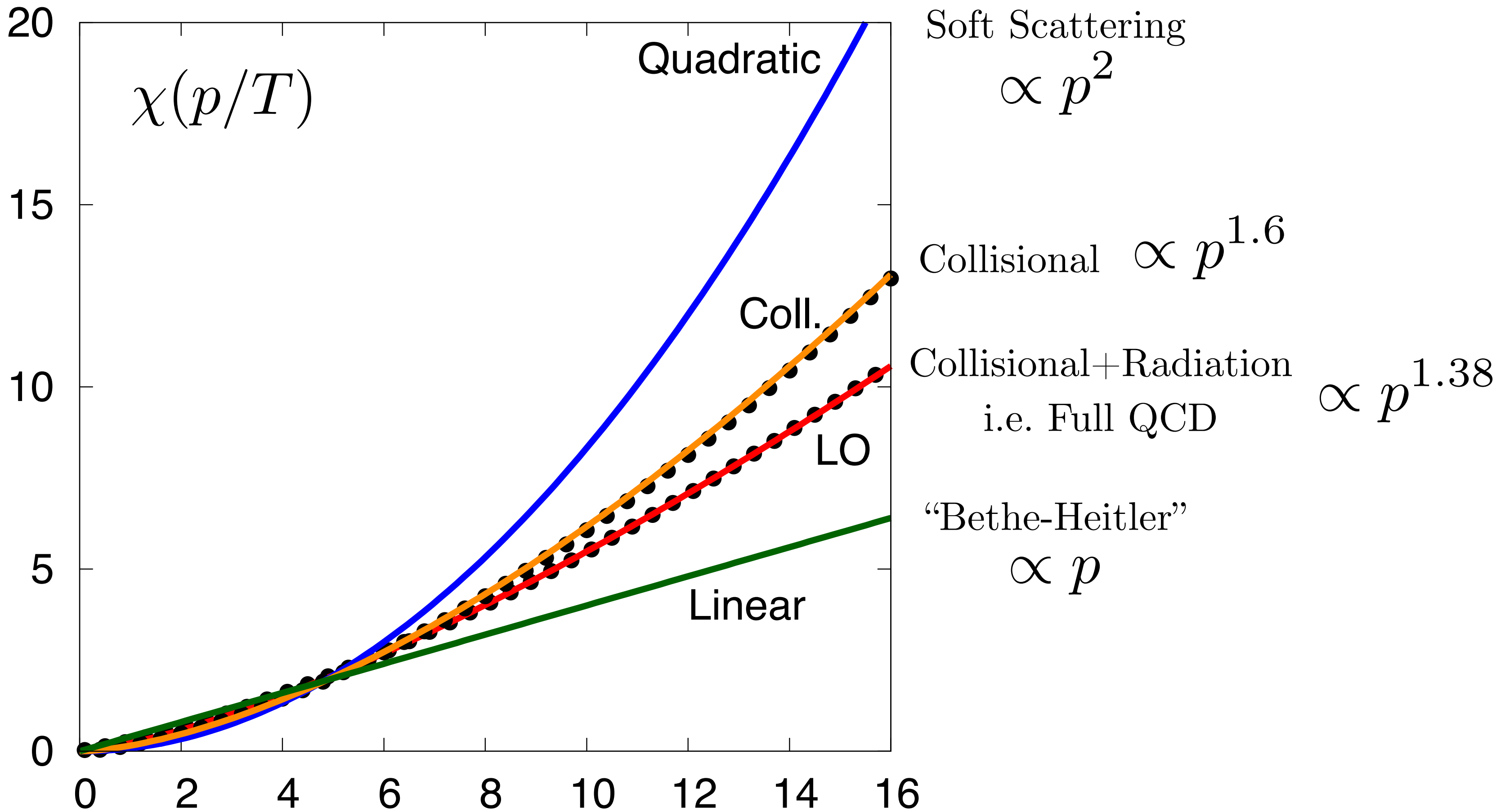
$$\frac{dp}{dt} \propto g^4 \log\left(\frac{p}{m_D}\right) \qquad \chi(p) \propto \frac{p^2}{\log p}$$

3. Radiative

$$\frac{\Delta p}{\Delta t} \propto g^2 \sqrt{\hat{q} E_p} \qquad \chi(p) \propto p^{3/2}$$

The forms of χ(p) at large momentum (including the constant) can be found analytically from the Boltzmann equation.

Weakly coupled pure-glue QCD



Results from numerical solution of linearized Boltzmann eqn.

Weakly coupled pure-glue QCD